

# Carathéodory's Theorem in Depth

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## Abstract

Let  $X$  be a finite set of points in  $\mathbb{R}^d$ . The Tukey depth of a point  $q$  with respect to  $X$  is the minimum number  $\tau_X(q)$  of points of  $X$  in a halfspace containing  $q$ . In this paper we prove a depth version of Carathéodory's theorem. In particular, we prove that there exist a constant  $c$  (that depends only on  $d$  and  $\tau_X(q)$ ) and pairwise disjoint sets  $X_1, \dots, X_{d+1} \subset X$  such that the following holds. Each  $X_i$  has at least  $c|X|$  points, and for every choice of points  $x_i$  in  $X_i$ ,  $q$  is a convex combination of  $x_1, \dots, x_{d+1}$ . We also prove depth versions of Helly's and Kirchberger's theorems.

## 1 Introduction

Carathéodory's theorem was proven by Carathéodory in 1907; it is one of the fundamental results in convex geometry. For sets of points in  $\mathbb{R}^d$ , it states the following.

**Theorem 1.1 (Carathéodory's theorem [9]).** *Let  $X$  be a set of points in  $\mathbb{R}^d$ . If a point  $q$  is contained in the convex hull,  $\text{Conv}(X)$ , of  $X$ , then there exist  $x_1, \dots, x_{d+1} \in X$  such that  $q \in \text{Conv}(\{x_1, \dots, x_{d+1}\})$ .*

The Tukey depth of a point  $q$  with respect to a finite point set  $X$  is a parameter  $0 \leq \tau_X(q) \leq 1$  that measures how “deep”  $q$  is inside of  $\text{Conv}(X)$ . In this paper we prove the following depth dependent version of Carathéodory's theorem.

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**Theorem 1.2 (Depth Carathéodory's theorem).** *Let  $X$  be a finite set of points in  $\mathbb{R}^d$  (or a Borel probability measure on  $\mathbb{R}^d$ ) and let  $q$  be a point in  $\mathbb{R}^d$  of positive Tukey depth with respect to  $X$ . Then there exists a positive constant  $c = c(d, \tau_X(q))$  (that depends only on  $d$  and  $\tau_X(q)$ ), such that the following holds. There exist pairwise disjoint subsets  $X_1, \dots, X_{d+1}$  of  $X$  such that for every choice of points  $x_i \in X_i$ ,  $q \in \text{Conv}(\{x_1, \dots, x_{d+1}\})$ . Moreover, each of the  $X_i$  consists of at least  $c|X|$  points.*

Informally, Theorem 1.2 states that the deeper  $q$  is inside  $\text{Conv}(X)$ , the larger subsets  $X_1, \dots, X_{d+1}$  of  $X$  exist, such that for every choice of points  $x_i \in X_i$ ,  $q$  is contained in the convex hull of  $\{x_1, \dots, x_{d+1}\}$ . In this sense, Carathéodory's theorem states that sets  $X_i$  of cardinality one always exist, whenever  $q$  has positive depth.

This paper is organized as follows. In Section 1.1 we review two other fundamental theorems in convex geometry: Helly's and Kirchberger's theorems. In Section 1.2 we formalize the notion of depth, and review centerpoint theorems that guarantee the existence of points of large depth. In Section 2 we present a new notion of depth together with its centerpoint theorem (Theorem 2.2). Using these results we prove the Depth Carathéodory's theorem (Theorem 1.2) in Section 3. In Section 3.1 we prove a stronger planar version of the Depth Carathéodory's theorem for points of small depth. Finally in Section 4, we prove depth versions of Helly's and Kirchberger's theorems.

## 1.1 Helly's and Kirchberger's Theorems

**Theorem 1.3 (Helly's theorem).** *Let  $\mathcal{F} = \{C_1, \dots, C_n\}$  be a family of  $n \geq d + 1$  convex sets in  $\mathbb{R}^d$ . Suppose that for every choice of  $(d + 1)$  sets  $D_i \in \mathcal{F}$  we have that  $\bigcap_{i=1}^{d+1} D_i \neq \emptyset$ . Then all of  $\mathcal{F}$  intersects, that is  $\bigcap \mathcal{F} \neq \emptyset$ .*

**Theorem 1.4 (Kirchberger's theorem [18]).** *Let  $R$  and  $B$  be finite sets of points in  $\mathbb{R}^d$ . Suppose that for every subset  $S \subset R \cup B$  of  $d + 2$  points, the set  $S \cap R$  can be separated from  $S \cap B$  by a hyperplane. Then  $R$  can be separated from  $B$  by a hyperplane.*

Helly's theorem was discovered in 1913 by Helly, but he did not published it until 1923 [13]. By then proofs by Radon [29] (1921) and by König [19] (1922) had been published. Carathéodory's and Helly's theorems are closely related in the sense that without much effort each can be proven assuming the other (see for example Eggleston's book [10]); and from either one of them one can prove Kirchberger's theorem. The link between Helly's and Carathéodory's theorems is given by the following lemma (see [10]).

**Lemma 1.5.** *Let  $x_1, \dots, x_n$  be points in  $\mathbb{R}^d$ ; let  $H_q(x_i)$  be the halfspace that does not contain  $q$ , and that is bounded by the following hyperplane: the hyperplane through  $x_i$  and perpendicular to the line passing through  $x_i$  and  $q$ . Then  $\bigcap_{i=1}^n H_q(x_i)$  is empty if and only if  $q \in \text{Conv}(\{x_1, \dots, x_n\})$ .*

*Proof.*

$\Rightarrow$ ) Suppose that  $q \notin \text{Conv}(\{x_1, \dots, x_n\})$ . Then there exist a hyperplane  $\Pi$  that separates  $q$  from  $\text{Conv}(\{x_1, \dots, x_n\})$ . Let  $\ell$  be the halfline with apex  $q$ , perpendicular to  $\Pi$ , and that intersects  $\Pi$ . Then for all  $H_q(x_i)$ , all but a finite segment of  $\ell$  is contained in  $H_q(x_i)$ . This implies that  $\bigcap_{i=1}^n H_q(x_i)$  is non-empty.

$\Leftarrow$ ) Suppose that  $q \in \text{Conv}(\{x_1, \dots, x_n\})$ . Then there exist  $\alpha_1, \dots, \alpha_n$ , such that every  $\alpha_i \geq 0$ ,

$$\sum_{i=1}^n \alpha_i = 1 \text{ and}$$

$$q = \sum_{i=1}^n \alpha_i x_i.$$

Note that for every  $y \in H_q(x_i)$ , we have that

$$(y - q) \cdot (x_i - q) \geq |q - x_i|^2 > 0.$$

Suppose to the contrary that there exists a point  $z \in \bigcap_{i=1}^n H_q(x_i)$ . Since at least one of the  $\alpha_i$  is greater than zero we have

$$\begin{aligned} 0 &= (z - q) \cdot (q - q) \\ &= (z - q) \cdot \left( \sum_{i=1}^n \alpha_i (x_i - q) \right) \\ &= \sum_{i=1}^n \alpha_i (z - q) \cdot (x_i - q) > 0, \end{aligned}$$

a contradiction.  $\square$

Given the close relationship between Carathéodory's and Helly's theorems, whenever there is a variant of one of them, it is natural to ask whether similar variants exist of the other. We mention their colorful and fractional variants.

In the classical versions of Helly's and Carathéodory's theorems, if every  $(d + 1)$ -tuple of a family of objects satisfies a certain property then all the family satisfies the property. In the statements of the colorful versions of these theorems, the objects are assigned one of  $d + 1$  colors. A *colorful tuple* is a tuple of objects containing one object of each available color. The hypothesis is that the colorful tuples satisfy the same property as in the classical versions and the conclusion is that one of the color classes satisfies the property. The colorful versions are the following.

**Theorem 1.6 (Colorful Carathéodory's theorem [4]).** *Let  $X$  be a finite set of points in  $\mathbb{R}^d$ , such that each point is assigned one of  $d + 1$  colors. Let  $q$  be a point in  $\mathbb{R}^d$  such that every colorful tuple of points of  $X$  does not contain  $q$  in its convex hull. Then one of the color classes of  $X$  does not contain  $q$  in its convex hull.*

**Theorem 1.7 (Colorful Helly’s theorem).** *Let  $\mathcal{F}$  be a finite family of convex sets in  $\mathbb{R}^d$ , such that each set is assigned one of  $d+1$  colors. Suppose that every colorful tuple of convex sets in  $\mathcal{F}$  has a non-empty intersection. Then a color class of  $\mathcal{F}$  has a non-empty intersection.*

The Colorful Helly’s theorem was first proved by Lovász (see [4]); the Colorful Carathéodory’s theorem was proved by Bárány. A stronger version of the Colorful Carathéodory theorem was proven independently by Holmsen, Pach, and Tverberg [15], and by Arocha, Bárány, Bracho, Fabila-Monroy and Montejano [3]. In that version the hypothesis is the same and the conclusion is that there exist two color classes such that  $q$  is not contained in the convex hull of their union. An even stronger version was proved by Meunier and Deza [24].

In the fractional versions, the hypothesis is that a large number of the  $(d+1)$ -tuples satisfies the same property as the classical version and the conclusion is that a large subfamily satisfies the property. The fractional version of Helly’s theorem is as follows.

**Theorem 1.8 (Fractional Helly’s theorem [17]).** *For every  $\alpha > 0$  and every dimension  $d \geq 1$  there exists a constant  $\beta = \beta(\alpha, d) > 0$  such that the following holds. Let  $\mathcal{F}$  be a family of  $n$  convex sets in  $\mathbb{R}^d$ . Suppose that at least  $\alpha \binom{n}{d+1}$  of the  $(d+1)$ -tuples of sets in  $\mathcal{F}$  has a non-empty intersection. Then there exists a subfamily  $\mathcal{F}' \subset \mathcal{F}$  of at least  $\beta n$  sets with a non-empty intersection.*

The Fractional Helly theorem was proved by Katchalski and Liu.

It is natural to ask whether a Fractional Carathéodory’s theorem exists. Interestingly enough, although the statement is true and well known, to the best of our knowledge there is no explicit mention of the Fractional Carathéodory’s theorem in the literature. The Fractional Carathéodory’s theorem is just the relationship between the Tukey and the simplicial depth defined in Section 1.2. The statement is the following.

**Theorem 1.9 (Fractional Carathéodory’s theorem).** *For every  $\alpha > 0$  and every dimension  $d \geq 1$  there exists a constant  $\beta = \beta(\alpha, d) > 0$  such that the following holds. Let  $X$  be a set of  $n$  points in  $\mathbb{R}^d$ . Suppose that  $q$  is a point such that at least  $\alpha \binom{n}{d+1}$  of the  $(d+1)$ -tuples of points do not contain  $q$  in their convex hull. Then there exists a subset  $X' \subset X$  of at least  $\beta n$  points such that its convex hull does not contain  $q$ .*

For recent variations and related results of Helly’s theorem, see the survey of Amenta, De Loera, and Soberón [2], and the survey by Holmsen and Wenger [14].

## 1.2 Centerpoint Theorems and Depth

Let  $X \subset \mathbb{R}^d$  be a set of  $n$  points. There are many formalizations of the notion of how deep a point  $q \in \mathbb{R}^d$  is inside  $\text{Conv}(X)$ . We use two of them in this paper; they are defined as follows.

- The *Tukey depth* of  $q$  with respect to  $X$ , is the minimum number of points of  $X$  in every closed halfspace that contains  $q$ ; we denote it by  $\tau_X(q)$ .

- The *simplicial depth* of  $q$  with respect to  $X$ , is the number of distinct subsets of  $d + 1$  points  $x_1, \dots, x_{d+1}$  of  $X$ , such that  $q$  is contained in  $\text{Conv}(\{x_1, \dots, x_{d+1}\})$ ; we denote it by  $\tilde{\sigma}_X(q)$ .

The word “simplicial” in the latter definition comes from the fact that  $q$  is contained in the simplex with vertices  $x_1, \dots, x_{d+1}$ . In other words,  $\tilde{\sigma}_X(q)$  is the number of simplices with vertices in  $X$  that contain  $q$  in their interior. The two definitions are not equivalent (one does not determine the other). They are however related; Afshani [1] has shown that  $\Omega(n\tilde{\tau}_X(q)^d) \leq \tilde{\sigma}_X(q) \leq O(n^d\tilde{\tau}_X(q))$ , and that these bounds are attainable; Wagner [32] has shown a tighter lower bound of

$$\tilde{\sigma}_X(q) \geq \frac{(d+1)\tilde{\tau}_X(q)^d n - 2d\tilde{\tau}_X(q)^{d+1}}{(d+1)!} - O(n^d).$$

Both definitions have generalizations to Borel probability measures on  $\mathbb{R}^d$ ; let  $\mu$  be such a measure.

- The *Tukey depth* of  $q$  with respect to  $\mu$ , is the minimum of  $\mu(H)$  over all closed halfspaces  $H$  that contain  $q$ ; we denote it by  $\tau_\mu(q)$ .
- The *simplicial depth* of  $q$  with respect to  $\mu$ , is the probability that  $q$  is in the convex hull of  $d + 1$  points chosen randomly and independently with distribution  $\mu$ ; we denote it by  $\sigma_\mu(q)$ .

The Tukey depth was introduced by Tukey [31] and the simplicial depth by Liu [20]. Both definitions aim to capture how deep a point is inside a data set. As a result they have been widely used in statistical analysis. For more information on these applications and other definitions of depth, see: The book edited by Liu, Serfling and Souvaine [21]; the survey by Rafalin and Souvaine [30]; and the monograph by Mosler [25].

For the purpose of this paper, we join the definitions for point sets and for probability measures by setting  $\tau_X(q) := \tilde{\tau}_X(q)/n$  and  $\sigma_X(q) := \tilde{\sigma}_X(q)/\binom{n}{d+1}$ . We refer to them as the Tukey and simplicial depth of point  $q$  with respect to  $X$ , respectively. (Alternatively, note that a Borel probability measure is obtained from  $X$  by defining the measure of an open set  $A \subset \mathbb{R}^d$  to be  $|A \cap X|/n$ .) Throughout the paper, for exposition purposes, we present the proofs of our results for sets of points rather than for Borel probability measures. However, we explicitly mention when such results also hold for Borel probability measures.

Long before the concept of Tukey depth came about (1975), a point of large Tukey depth was shown to always exist. This was first proved in the plane by Neumann [26] in 1945. Using Helly’s theorem, Rado [28] generalized this result to higher dimensions in 1947. We rephrase this theorem in terms of the Tukey depth as follows.

**Theorem 1.10 (Centerpoint theorem for Tukey depth).** *Let  $X$  be a finite set of points in  $\mathbb{R}^d$  (or a Borel probability measure on  $\mathbb{R}^d$ ). Then there exists a point  $q$  such that  $\tau_X(q) \geq \frac{1}{d+1}$ .*

A point satisfying Theorem 1.10 is called a *centerpoint*. The bound of Theorem 1.10 is tight; there exist sets of  $n$  points in  $\mathbb{R}^d$  such that no point of  $\mathbb{R}^d$  has Tukey depth larger than  $\frac{1}{d+1}$  with respect to these point sets.

The preceding of a centerpoint theorem before the definition of Tukey depth also occurred with the definition of simplicial depth (1990). Boros and Füredi [7] proved in 1984 that if  $X$  is a set of  $n$  points in the plane, then there exists a point  $q$  contained in at least  $\frac{2}{9}\binom{n}{3}$  of the triangles with vertices on  $X$ . This was generalized to higher dimensions by Bárány [4] in 1982. This result also holds for Borel probability measures (see Wagner’s PhD thesis [32]); we rephrase it in terms of the simplicial depth as follows.

**Theorem 1.11 (Centerpoint theorem for simplicial depth).** *Let  $X$  be a finite set of points in  $\mathbb{R}^d$  (or a Borel probability measure on  $\mathbb{R}^d$ ). Then there exists a point  $q$  and a constant  $c_d > 0$  (depending only on  $d$ ) such that  $\sigma_X(q) \geq c_d$ .*

Theorem 1.11 has been named the “First Selection Lemma” by Matoušek [22]. In contrast with Theorem 1.10 the exact value of  $c_d$  (for values of  $d$  greater than 2) is far from known. In the case of when  $X$  is a point set, the search for better bounds for  $c_d$  has been an active area of research. The current best upper bound (for point sets) is  $c_d \leq \frac{(d+1)!}{(d+1)^{d+1}}$ . This was proven by Bukh, Matoušek and Nivasch [8]; they conjecture that this is the exact value of  $c_d$ . As for the lower bound Bárány’s original proof yields  $c_d \geq \frac{1}{(d+1)^{d+1}}$ . Using topological methods Gromov [12] has significantly improved this bound to  $c_d \geq \frac{2d}{(d+1)!(d+1)}$ . Shortly after, Karashev [16] provided a simpler version of Gromov’s proof, but still using topological methods. Matoušek and Wagner [23] gave an expository account of the combinatorial part of Gromov’s proof; they also slightly improved Gromov’s bound and showed limitations on his method.

## 2 Projection Tukey Depth

Along the way of proving the Depth Carathéodory’s theorem we prove a result (Theorem 2.2) similar in spirit to the centerpoint theorems above. Taking a lesson from history we first define a notion of depth and then phrase our result accordingly.

Assume that  $d \geq 2$  and let  $X$  be a set of  $n$  points in  $\mathbb{R}^d$ . Let  $q$  be a point of  $\mathbb{R}^d \setminus X$ . Given a point  $p \in \mathbb{R}^d$  distinct from  $q$ , let  $r(p, q)$  be the infinite ray passing through  $p$  and with apex  $q$ . For a set  $A \subset \mathbb{R}^d$ , not containing  $q$ , let  $R(A, q) := \{r(p, q) : p \in A\}$  be the set of rays with apex  $q$  and passing through a point of  $A$ . Let  $\Pi$  be an oriented hyperplane containing  $q$ , and let  $\Pi^+$  and  $\Pi^-$  be two hyperplanes, parallel to  $\Pi$ , strictly above and below  $\Pi$ , respectively. Let  $X^+ := \Pi^+ \cap R(X, q)$  and  $X^- := \Pi^- \cap R(X, q)$ . Let  $L(q)$  be the set of straight lines that contain  $q$ . See Figure 1.

We define the *projection Tukey depth* of  $q$  with respect to  $\Pi$  and  $X$ , to be

$$\pi_{X, \Pi}(q) := \max\{\min\{\tau_{X^+}(\ell \cap \Pi^+), \tau_{X^-}(\ell \cap \Pi^-)\} : \ell \in L(q)\}.$$

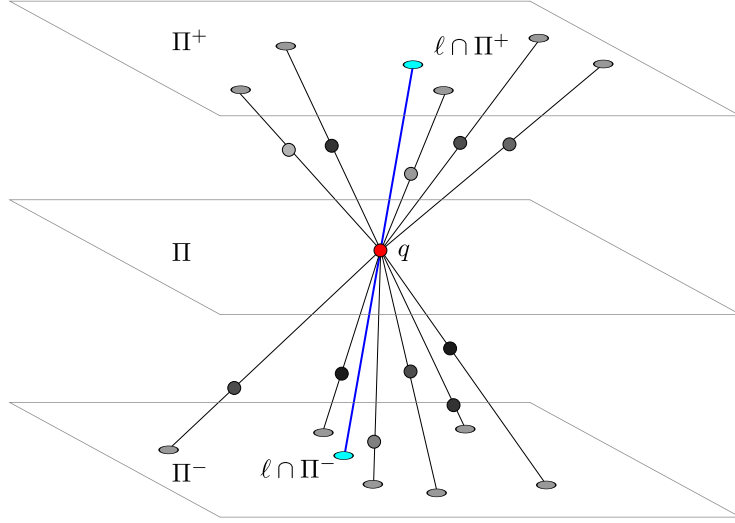


Figure 1: Depiction of  $q$ ,  $X$ ,  $X^+$ ,  $X^-$ ,  $\Pi$ ,  $\Pi^+$ , and  $\Pi^-$ .

Intuitively if  $q$  has large projection Tukey depth with respect to  $\Pi$  and  $X$ , then there exists a direction in which  $q$  can be projected to  $\Pi^+$  and  $\Pi^-$ , such that the images of  $q$  have large Tukey depth with respect to  $X^+$  and  $X^-$ . See Figure 1. Finally we define the *projection Tukey depth* of  $q$  with respect to  $X$ , as the minimum of this value over all  $\Pi$ . That is,

$$\pi_X(q) := \min\{\pi_{X,\Pi}(q) : \Pi \text{ is a hyperplane containing } q\}.$$

To provide the definition for Borel probability measures, we use  $\mu$  to define Borel measures  $\mu^+$  and  $\mu^-$  on  $\Pi^+$  and  $\Pi^-$ , respectively. Let  $U$  and  $D$  be the set of points of  $\mathbb{R}^d$  that lie above and below  $\Pi$ , respectively. Let  $\mu^+(A) := \mu(R(A, q))/\mu(U)$  for sets  $A \subset \Pi^+$ , and  $\mu^-(A) := \mu(R(A, q))/\mu(D)$  for sets  $A \subset \Pi^-$ . The *projection Tukey depth* of  $q$  with respect to  $\Pi$  and  $\mu$ , and the *projection Tukey depth* of  $q$  with respect to  $\mu$  are defined respectively as

$$\pi_{\mu,\Pi}(q) := \max\{\min\{\tau_{\mu^+}(\ell \cap \Pi^+), \tau_{\mu^-}(\ell \cap \Pi^-)\} : \ell \in L(q)\} \text{ and}$$

$$\pi_\mu(q) := \min\{\pi_{\mu,\Pi}(q) : \Pi \text{ is a hyperplane containing } q\}.$$

The following lemma lower bounds the projection Tukey depth of  $q$  in terms of its Tukey depth.

**Lemma 2.1.** *Let  $X$  be a set of  $n$  points in  $\mathbb{R}^d$  (or a Borel probability measure on  $\mathbb{R}^d$ ) and let  $q$  be a point in  $\mathbb{R}^d$ . Then  $\pi_X(q) \geq \min\{\tau_X(q), \frac{1}{d}\}$ .*

*Proof.* Let  $\delta := \min\{\tau_X(q), \frac{1}{d}\}$  and  $\varepsilon > 0$ . We prove the result by showing that  $\pi_X(q) \geq \delta - \varepsilon$ . Let  $\Pi$  be a hyperplane containing  $q$ ; define  $\Pi^+$ ,  $\Pi^-$ ,  $X^+$  and

$X^-$  with respect to  $\Pi$ , as above. We look for a straight line  $\ell$  passing through  $q$  such that

$$\tau_{X^+}(\ell \cap \Pi^+) \geq \delta - \varepsilon \text{ and } \tau_{X^-}(\ell \cap \Pi^-) \geq \delta - \varepsilon.$$

For this we find a point  $q^+$  in  $\Pi^+$ , with certain properties, such that the straight line passing through  $q$  and  $q^+$  is the desired  $\ell$ .

For each point  $p \in X^-$ , let  $p'$  be intersection point of the line passing through  $q$  and  $p$  with  $\Pi^+$ ; let  $X'$  be the set of all such  $p'$ . Note that  $X'$  is just the reflection of  $X^-$  through  $q$  into  $\Pi^+$ . Let  $\mathcal{P}_q$  be the set of hyperplanes passing through  $q$  and not parallel to  $\Pi$ , and let  $\mathcal{P}_+$  be the set of  $(d-2)$ -dimensional flats contained in  $\Pi^+$ . There is a natural one-to-one correspondence between  $\mathcal{P}_+$  and  $\mathcal{P}_q$ . For each  $l \in \mathcal{P}_+$ , let  $\pi_l$  be the hyperplane in  $\mathcal{P}_q$  containing both  $q$  and  $l$ ; conversely for each  $\pi \in \mathcal{P}_q$ , let  $l_\pi$  be the  $(d-2)$ -dimensional flat in  $\mathcal{P}_+$  defined by the intersection of  $\pi$  and  $\Pi^+$ .

Note that the following relationship holds for every pair of points  $p_1 \in X^+$  and  $p_2 \in X^-$ ;

$$\begin{aligned} p_1 \text{ and } p_2 \text{ are on the same half-space defined by a } \pi \in \mathcal{P}_q \text{ if and} \\ \text{only if } p_1 \text{ and } p'_2 \text{ are in the opposite half-spaces of } \Pi^+ \text{ defined} \\ \text{by } l_\pi. \end{aligned} \tag{1}$$

We use this observation to find  $q^+$ .

Let  $\mathcal{H}$  be the set of all  $(d-1)$ -dimensional half-spaces of  $\Pi^+$  that contain more than  $|X^+| - (\delta - \varepsilon)|X^+|$  points of  $X^+$ ; let  $\mathcal{H}'$  be the set of all  $(d-1)$ -dimensional half-spaces of  $\Pi^+$  that contain more than  $|X'| - (\delta - \varepsilon)|X'|$  points of  $X'$ . Therefore, since  $\delta - \varepsilon < \frac{1}{d}$ , a centerpoint of  $X^+$ , given by Theorem 1.10, is contained in every halfspace in  $\mathcal{H}$ ; otherwise, we obtain a contradiction since the opposite half-space would contain the centerpoint and less than  $\frac{1}{d}|X^+|$  points of  $X^+$ . Likewise a centerpoint of  $X'$ , given by Theorem 1.10, is contained in every halfspace in  $\mathcal{H}'$ . Therefore,  $Q := \bigcap \mathcal{H}$  and  $Q' := \bigcap \mathcal{H}'$  are non-empty. A point in the intersection of  $Q$  and  $Q'$  is our desired  $q^+$ .

For the sake of a contradiction, suppose that  $Q$  and  $Q'$  are disjoint. Let  $l \in \mathcal{P}_+$  be a  $(d-2)$ -dimensional flat that separates them in  $\Pi^+$ . Let  $h$  be the halfspace (in  $\mathbb{R}^d$ ) defined by  $\pi_l$  that contains  $Q'$  and does not contain  $Q$ . Note that  $h$  contains at least  $|X'| - (\delta - \varepsilon)|X'|$  points of  $X'$  and at most  $(\delta - \varepsilon)|X^+|$  points of  $X^+$ . By (2),  $h$  contains at most  $(\delta - \varepsilon)|X^-|$  points of  $X^-$ . Therefore,  $h$  contains at most  $(\delta - \varepsilon)|X^+| + (\delta - \varepsilon)|X^-| = (\delta - \varepsilon)|X|$  points of  $X$ —a contradiction. Therefore,  $Q$  and  $Q'$  intersect.

Let  $q^+$  be a point in  $Q \cap Q'$  and let  $\ell$  be the straight line passing through  $q$  and  $q^+$ ; let  $q^- := \ell \cap \Pi^-$ . Note that  $q^-$  is in the intersection of all  $(d-1)$ -dimensional halfspaces that contain more than  $|X^-| - (\delta - \varepsilon)|X^-|$  points of  $X^-$ . We have that every halfspace in  $\Pi^+$  that contains  $q^+$ , contains at least  $(\delta - \varepsilon)|X^+|$  points of  $X^+$ , and every half space in  $\Pi^-$  that contains  $q^-$ , contains at least  $(\delta - \varepsilon)|X^-|$  points of  $X^-$ ; the result follows.  $\square$

Although Lemma 2.1 bounds the projection Tukey depth of  $q$  with respect to  $X$  in terms of its Tukey depth, it does so up to a point; when the Tukey depth



is larger than  $\frac{1}{d}$ , the best lower bound on the projection Tukey depth given by Lemma 2.1 is of  $\frac{1}{d}$ ; this bound can be tight. Suppose that  $X$  is such that  $X^-$  is equal to the reflection of  $X^+$  through  $q$  into  $\Pi^-$ . Moreover, assume that  $X^+$  is such that every point in  $\Pi^+$  has Tukey depth of at most  $\frac{1}{d}$  with respect to  $X^+$ . Note that in this case the Tukey depth of  $q$  with respect to  $X$  is  $\frac{1}{2}$  and the projection Tukey depth of  $q$  with respect to  $\Pi$  and  $X$  is at most  $\frac{1}{d}$ . The latter implies that the projection Tukey depth of  $q$  with respect to  $X$  is at most  $\frac{1}{d}$ .

Lemma 2.1 and Theorem 1.10 yield at once a centerpoint theorem for the projection Tukey depth.

**Theorem 2.2 (Centerpoint theorem for projection Tukey depth).** *Let  $X$  be a finite set of points in  $\mathbb{R}^d$  (or a Borel probability measure on  $\mathbb{R}^d$ ). Then there exists a point  $q$  such that  $\pi_X(q) \geq \frac{1}{d+1}$ .*

*Proof.* By Theorem 1.10 there exists a point  $q$  of Tukey depth with respect to  $X$  of at least  $\frac{1}{d+1}$ . By Lemma 2.1,  $\pi_X(q) \geq \min\{\tau_X(q), \frac{1}{d}\} \geq \frac{1}{d+1}$ .  $\square$

### 3 Depth Carathéodory's Theorem

Using Lemma 2.1 it can be shown by induction on  $d$  that if  $q$  has a large Tukey depth with respect to  $X$  and  $|X|$  is sufficiently large with respect to  $d$ , then there exist large subsets  $X_1, \dots, X_{2^d}$  of  $X$ , such that for every choice of points  $x_i \in X_i$ ,  $q$  is contained in  $\text{Conv}(\{x_1, \dots, x_{2^d}\})$ . To reduce this number of subsets to  $d+1$  we need a result from [6].

The order type is a combinatorial abstraction of the geometric properties of point sets. They were introduced by Goodman and Pollack in [11]. Two sets of points  $X$  and  $X'$  in  $\mathbb{R}^d$  are said to have the same *order type* if there is a bijection,  $\varphi$ , between them that satisfies the following. The orientation of every  $(d+1)$ -tuple  $(x_1, \dots, x_{d+1})$  of points of  $X$  is equal to the orientation of the corresponding  $(d+1)$ -tuple  $(\varphi(x_1), \dots, \varphi(x_{d+1}))$  of  $X'$ . This means that the signs of the determinants  $\det \left[ \begin{pmatrix} x_1 \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} x_{d+1} \\ 1 \end{pmatrix} \right]$  and  $\det \left[ \begin{pmatrix} \varphi(x_1) \\ 1 \end{pmatrix}, \dots, \begin{pmatrix} \varphi(x_{d+1}) \\ 1 \end{pmatrix} \right]$  are equal. Let  $\mathbf{x} := (x_1, \dots, x_m)$  and  $\mathbf{y} := (y_1, \dots, y_m)$  be two  $m$ -tuples of  $\mathbb{R}^d$  (for  $m \geq d+1$ ). We say that  $\mathbf{x}$  and  $\mathbf{y}$  have the same order type if for every subsequence  $(i_1, \dots, i_{d+1})$  of  $(1, \dots, m)$ , the orientation of the  $(d+1)$ -tuple  $(x_{i_1}, \dots, x_{i_{d+1}})$  of  $\mathbf{x}$  is the same as the orientation of the  $(d+1)$ -tuple  $(y_{i_1}, \dots, y_{i_{d+1}})$  of  $\mathbf{y}$ .

In particular this implies that if a point  $q \in \mathbb{R}^d$  is such that  $\mathbf{x} := (x_1, \dots, x_m, q)$  and  $\mathbf{x}' := (x'_1, \dots, x'_m, q)$  have the same order type, then  $q \in \text{Conv}(\{x_1, \dots, x_m\})$  if and only if it  $q \in \text{Conv}(\{x'_1, \dots, x'_m\})$ . Bárány and Valtr [6] proved the following theorem on order types of tuples of point sets.

**Theorem 3.1 (Same-type lemma).** *For any integers  $d, m \geq 1$ , there exists a constant  $c' = c'(d, m) > 0$  (that depends only on  $d$  and  $m$ ) such that the following holds. Let  $X_1, X_2, \dots, X_m$  be finite sets of points in  $\mathbb{R}^d$  (or Borel probability measures on  $\mathbb{R}^d$ ). Then there exist  $Y_1 \subseteq X_1, \dots, Y_m \subseteq X_m$ , such*

that every pair of  $m$ -tuples  $(z_1, \dots, z_m)$  and  $(z'_1, \dots, z'_m)$  with  $z_i, z'_i \in Y_i$  have the same order type. Moreover for all  $i = 1, 2, \dots, m$ ,  $|Y_i| \geq c'|X_i|$ .

We note that the Same-type lemma is phrased only for points in general position in both [6] and in Matoušek's book [22]. However, in Remark 5 of [6] it is noted that the result holds for Borel probability measures and for points not in general position.

We are ready to prove the Depth Carathéodory's theorem.

*Proof of Theorem 1.2.* The result holds for  $d = 1$  with  $c(1, \tau_X(q)) = \tau_X(q)$ . Assume that  $d > 1$  and proceed by induction on  $d$ . Let  $n := |X|$  and let  $\Pi$  be a hyperplane containing  $q$  that bisects  $X$ . This is, on both of the open halfspaces defined by  $\Pi$  there are at least  $\lfloor n/2 \rfloor$  points of  $X$ . Define  $X^+, X^-, \Pi^+$  and  $\Pi^-$  as in Section 2 with respect to  $X$  and  $\Pi$ .

Let  $\delta := \min\{\tau_X(q), \frac{1}{d}\}$ . By Lemma 2.1 the projection Tukey depth of  $q$  with respect to  $X$  is at least  $\delta$ . Therefore, there exist a line  $\ell$  such that  $q^+ := \ell \cap \Pi^+$  and  $q^- := \ell \cap \Pi^-$  have Tukey depth at least  $\delta$  with respect to  $X^+$  and  $X^-$  respectively. By induction there exists a constant  $c(d-1, \delta)$  and sets  $Y_1^+, \dots, Y_d^+ \subset X^+$  and  $Y_1^-, \dots, Y_d^- \subset X^-$  such that the following holds. Every  $Y_i^+$  has cardinality at least  $c(d-1, \delta)|X^+| \geq c(d-1, \delta)\lfloor n/2 \rfloor$ , and every  $Y_i^-$  has cardinality at least  $c(d-1, \delta)|X^-| \geq c(d-1, \delta)\lfloor n/2 \rfloor$ ; moreover,  $q^+ \in \text{Conv}(\{x_1, \dots, x_d\})$  for every choice of  $x_i \in Y_i^+$ , and  $q^- \in \text{Conv}(\{x'_1, \dots, x'_d\})$  for every choice of  $x'_i \in Y_i^-$ .

Therefore,  $q$  is in the convex hull of  $\{x_1, \dots, x_d\} \cup \{x'_1, \dots, x'_d\}$  for every choice of  $x_i \in Y_i^+$  and  $x'_i \in Y_i^-$ . (2)

Apply the Same-type lemma to  $Y_1^+, \dots, Y_d^+, Y_1^-, \dots, Y_d^-, \{q\}$ , and obtain sets  $Z_1 \subset Y_1^+, \dots, Z_d \subset Y_d^+$ , and  $Z_{d+1} \subset Y_1^-, \dots, Z_{2d} \subset Y_d^-$  each of at least  $c'(d, 2d)c(d-1, \delta)\lfloor n/2 \rfloor$  points, such that the following holds. Every pair of  $(2d+1)$ -tuples  $(z_1, \dots, z_{2d}, q)$  and  $(z'_1, \dots, z'_{2d}, q)$  with  $z_i \in Z_i$  and  $z'_i \in Z'_i$  have the same order type. Let  $(z_1, \dots, z_{2d}, q)$  be one such  $(2d+1)$ -tuple. By (2),  $q$  is in the convex hull of  $\{z_1, \dots, z_{2d}\}$ . Therefore, by Carathéodory's theorem there exists a  $(d+1)$ -tuple  $(i_1, \dots, i_{d+1})$  such that  $q$  is a convex combination of  $z_{i_1}, \dots, z_{i_{d+1}}$ . The result follows by setting  $X_j := Z_{i_j}$ . □

Note that Theorem 1.2 also applies for the simplicial depth. That is, suppose that  $q$  is in a constant proportion of the simplices spanned by points of  $X$ . Then, there exist subsets  $X_1, \dots, X_{d+1}$  of  $X$ , of linear size, such that  $q$  is in every simplex that has exactly one vertex in each  $X_i$ .

It is noted in [6] that the constant in Theorem 3.1 is bounded from below by

$$c'(d, m) \geq (d+1)^{-(2^d-1)\binom{m-1}{d}}.$$

Therefore, the proof of Theorem 1.2 implies that  $c(d, \tau_X(q))$  is an increasing function on  $\tau_X(q)$ , when  $0 < \tau_X(q) \leq \frac{1}{d}$  and  $d$  fixed.

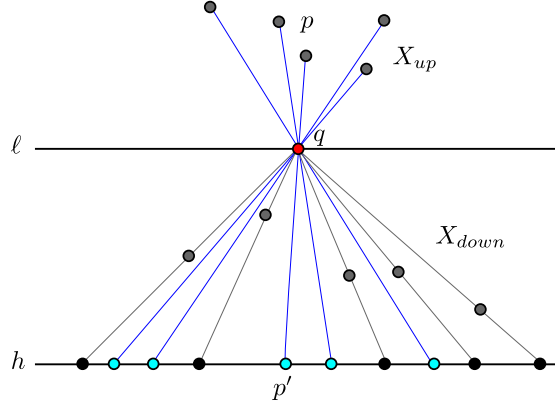


Figure 2: Illustration of the projection in the proof of Theorem 3.2.  $X'_{up}$  is represented as blue points on the line  $h$ , and  $X'_{down}$  as black points.

### 3.1 Depth Carathéodory's Theorem in the Plane

The Depth Carathéodory's theorem (Theorem 1.2) can be applied when  $q$  has constant Tukey depth with respect to  $X$ . That is when  $\tau_X(q) = c$  for some positive constant  $c$ . In this section we prove the Depth Carathéodory's theorem in the plane for points of subconstant depth with respect to  $X$ , for example when  $\tau_X(q) = \frac{1}{n}$ . We use the simplicial depth as it is easier to quantify the depth of a point in this case. Also, we revert to using the simplicial depth of  $q$  with respect to  $X$  as the number of triangles  $\tilde{\sigma}_X(q)$  with vertices on  $X$  that contain  $q$  (rather than this number divided by  $\binom{n}{2}$ ). We consider only the case when  $X$  is a set of  $n$  points in general position in the plane.

**Theorem 3.2.** *Let  $X$  be a set of  $n$  points in general position in the plane. Let  $q \in \mathbb{R}^2 \setminus X$  be a point such that  $X \cup \{q\}$  is in general position. Then  $X$  contains three disjoint subsets  $X_1, X_2, X_3$  such that  $|X_1||X_2||X_3| \geq \frac{1}{16 \ln n} \tilde{\sigma}_X(q)$ , and  $q$  is contained in every triangle with vertices  $x \in X_1, y \in X_2, z \in X_3$ .*

*Proof.* Let  $\ell$  be a line passing through  $q$  that bisects  $X$ . Without loss of generality, assume that at least half of the triangles which contain  $q$  and which have their vertices in  $X$ , have two of their vertices below  $\ell$ ; denote this set of triangles by  $T$ . Further assume that  $\ell$  is horizontal. Let  $X_{down}$  be the points of  $X$  below  $\ell$  and let  $X_{up}$  be the points of  $X$  above  $\ell$ . Project  $X$  on a horizontal line  $h$  far below  $X_{down}$  as follows. The image  $p'$  of a point  $p \in X$  is the intersection point of the line through  $q$  and  $p$  with  $h$ . Let  $X'_{up}$  and  $X'_{down}$  be the images of  $X_{up}$  and  $X_{down}$ , respectively. See Figure 2.

Note that a triangle with vertices  $x \in X_{down}, y \in X_{up}$  and  $z \in X_{down}$  contains  $q$  if and only if  $x', y'$  and  $z'$  appear in the order  $(x', y', z')$  with respect to  $h$ . For a point  $p' \in X'_{up}$  let  $l(p')$  be the number of points in  $X'_{down}$  to its left, and let  $r(p')$  be the number of points in  $X'_{down}$  to its right. By the previous

observations we have that

$$|T| = \sum_{p' \in X'_{up}} l(p')r(p').$$

Let  $\mathcal{L} := \{p' \in X'_{up} : r(p') \geq l(p')\}$  and  $\mathcal{R} := \{p' \in X'_{up} : r(p') < l(p')\}$ . Assume without loss of generality that at least half of the triangles in  $T$  are such that one of its vertices lies in  $\mathcal{L}$ . That is

$$\sum_{p' \in \mathcal{L}} r(p')l(p') \geq \frac{|T|}{2}. \quad (3)$$

Also note that

$$\sum_{p' \in \mathcal{L}} r(p')l(p') \leq n \sum_{p' \in \mathcal{L}} l(p'). \quad (4)$$

Let  $p'_1, \dots, p'_m$  be the points in  $\mathcal{L}$  in their left-to-right order in  $h$ . Consider the sum

$$\sum_{i=1}^m l(p'_i)(m-i+1).$$

Let  $M$  be the maximum value attained by a term in this sum. Then, for all  $i = 1, \dots, m$ , we have that  $l(p'_i) \leq M \frac{1}{(m-i+1)}$ . Combining this observation with (3) and (4), we have

$$M \ln n \geq M \sum_{i=1}^m \frac{1}{(m-i+1)} \geq \sum_{i=1}^m l(p'_i) \geq \frac{|T|}{2n}.$$

Therefore,

$$M \geq \frac{|T|}{2n \ln n}. \quad (5)$$

Let  $i^*$  be such that  $l(p'_{i^*})(m-i^*+1) = M$ . Set:  $X_1$  to be the set of points of  $X_{down}$  such their images are to the left of  $p_{i^*}$ ,  $X_2$  to be set of points  $\{p_{i^*}, \dots, p_m\}$  of  $X_{up}$  whose images lie between  $p'_{i^*}$  and  $p'_m$ , and  $X_3$  the set of set of points of  $X_{down}$  such their images are to the right of  $p'_m$ . Note that every triangle with vertices  $x \in X_1$ ,  $y \in X_2$ ,  $z \in X_3$ , contains  $q$ . Also note that  $|X_{down}| \geq n/2$  and  $r(p') \geq l(p')$  for all  $p' \in \mathcal{L}$  imply that  $|X_3| \geq n/4$ . Moreover,

$$|X_1||X_2||X_3| \geq l(p'_{i^*})(m-i^*+1) \frac{n}{4} = M \frac{n}{4} \geq \frac{1}{8 \ln n} |T| \geq \frac{1}{16 \ln n} \tilde{\sigma}_X(q);$$

the result follows.  $\square$

## 4 Helly's and Kirchberger's Depth Theorems

In this section we use the Depth Carathéodory's theorem to prove “depth” versions of Helly's and Kirchberger's theorems. Afterwards, for the sake of completeness we show that the Depth Helly's theorem implies the Depth Carathéodory theorem for point sets.

**Theorem 4.1 (Depth Helly's theorem).** *Let  $0 \leq \beta \leq 1$  and  $\alpha = c(d, 1 - \beta)$  (where  $c(d, \beta)$  is as in Theorem 1.2). Let  $\mathcal{F} = \{C_1, \dots, C_n\}$  be a family of  $n \geq d + 1$  convex sets in  $\mathbb{R}^d$ . Suppose that for every choice of subfamilies  $F_1, \dots, F_{d+1}$  of  $\mathcal{F}$ , each with at least  $\alpha n$  sets, there exists a choice of sets  $D_i \in F_i$  such that  $\bigcap_{i=1}^{d+1} D_i \neq \emptyset$ . Then there exists a subfamily  $\mathcal{F}' \subset \mathcal{F}$  of at least  $\beta n$  sets such that  $\bigcap \mathcal{F}' \neq \emptyset$ .*

*Proof.* We assume that the  $n$  convex sets are compact. The result for non-compact convex sets follows easily. Suppose for a contradiction that every subfamily of  $\mathcal{F}$  of at least  $\beta n$  sets is not intersecting. For every  $\mathcal{F}'$  subfamily of  $\mathcal{F}$  of at least  $\beta n$  sets and every  $x \in \mathbb{R}^d$  define

$$f(x, \mathcal{F}') = \max\{d(x, C) : C \in \mathcal{F}'\},$$

where  $d(x, C)$  denotes the distance from point  $x$  to the set  $C$ . By hypothesis  $f(x, \mathcal{F}') > 0$ . Since the elements of  $\mathcal{F}$  are closed and bounded,  $f(x, \mathcal{F}')$  attains a minimum value at some point of  $\mathbb{R}^d$ . Let

$$f(x) = \min\{f(x, \mathcal{F}') : \mathcal{F}' \subset \mathcal{F} \text{ and } |\mathcal{F}'| \geq \beta n\}$$

Since there are a finite number of subfamilies of  $\mathcal{F}$  and the previous remark,  $f(x)$  attains its minimum value at some point of  $q \in \mathbb{R}^d$ .

For every  $C_i \in \mathcal{F}$ , let  $x_i$  be a point in  $C_i$  such that  $d(q, C_i) = d(q, x_i)$ . Let  $X$  be the multiset with elements  $x_1, \dots, x_n$ . Let  $C'_1, \dots, C'_k$  be the sets in  $\mathcal{F}$  such that  $f(q) = d(q, C'_i)$  and let  $x'_1, \dots, x'_k$  be their corresponding points in  $X$ . We first show that

$$q \text{ is contained in } \text{Conv}(\{x'_1, \dots, x'_k\}). \quad (6)$$

Suppose for a contradiction that  $q$  is not contained in  $\text{Conv}(\{x'_1, \dots, x'_k\})$ . Let  $q'$  be a point closer to  $\text{Conv}(\{x'_1, \dots, x'_k\})$  than  $q$ . Let  $\mathcal{F}'$  be a subfamily of  $\mathcal{F}$  of at least  $\beta n$  sets such that  $f(q, \mathcal{F}') = f(q)$ . Suppose that  $C$  is a set of  $\mathcal{F}'$  such that  $d(q, C) < f(q, \mathcal{F}') = f(q)$ . Note that we can choose  $q'$  sufficiently close to  $q$  so that  $d(q', C) < f(q)$ . We choose  $q'$  sufficiently close to  $q$  so that  $d(q', C) < f(q)$  for all such  $C \in \mathcal{F}'$ . Let now  $C$  be a set of  $\mathcal{F}$  such that  $d(q, C) = f(q, \mathcal{F}') = f(q)$ . Since the closest point of  $C$  to  $q$  is one of the  $x'_i$ , then  $d(q', C) < d(q, C)$ . Therefore,  $f(q', \mathcal{F}') < f(q, \mathcal{F}')$  and  $f(q') < f(q)$ —a contradiction to our choice of  $q$ .

To every point  $x_i \in X$  assign the weight  $\frac{m(x_i)}{n}$ , where  $m(x_i)$  is the multiplicity of  $x_i$  in  $X$ . Thus, we may regard  $X$  as a Borel probability measure. We now show that

$$q \text{ has Tukey depth greater than } 1 - \beta \text{ with respect to } X. \quad (7)$$

Suppose to the contrary that there exists a hyperplane  $\Pi$  through  $q$ , such that the measure of one of the two halfspaces bounded by  $\Pi$  is at most  $(1 - \beta)$ . Then on the opposite side there is a subset  $X'$  of  $X$  of measure at least  $\beta$ . By (6) at least one of these points must be in  $\{x'_1, \dots, x'_k\}$ . If we move  $q$  slightly closer to  $\text{Conv}(X')$  we obtain a  $q'$  with  $f(q') < f(q)$ ; this is a contradiction to the choice of  $q$ .

Apply the Depth Carathéodory theorem to  $X$  and  $q$ , and obtain  $X_1, \dots, X_{d+1}$  subsets of  $X$  each of at least  $\alpha n$  points, such that the following holds. For every choice of  $y_i$  in  $X_i$ ,  $q$  is contained in  $\text{Conv}(\{y_1, \dots, y_{d+1}\})$ . Let  $C''_i$  be the set in  $\mathcal{F}$  that defines  $y_i$ . Then by Lemma 1.5,

$$\bigcap_{i=1}^{d+1} C''_i = \emptyset.$$

Let

$$F_i = \{C_j \in \mathcal{F} : \text{there is a point } y \text{ in } X_i \text{ defined by } C_j\}$$

Since each  $X_i$  is a multiset and every  $x'_i$  is associated to a different  $C_i$ , then  $F_1, \dots, F_{d+1}$  are subsets of  $\mathcal{F}$ , each with at least  $\alpha n$  sets, such that for every choice  $D_i \in F_i$ ,  $\bigcap_{i=1}^{d+1} D_i$  is empty—a contradiction.  $\square$

*Proof of Theorem 1.2 (for point sets) using Theorem 4.1.*

Let  $X := \{x_1, \dots, x_n\}$  be a set of  $n$  points in  $\mathbb{R}^d$  and let  $q$  be a point in  $\mathbb{R}^d$ . Let  $\beta > 1 - \tau_X(q)$ . Define  $H_q(x_i)$  as in Lemma 1.5; we say that  $H_q(x_i)$  is defined by  $x_i$ . Let  $\mathcal{F} := \{H_q(x_1), \dots, H_q(x_n)\}$ .

Let  $\mathcal{F}'$  be a subfamily of  $\mathcal{F}$  of  $\beta n$  sets. Let  $X'$  be the subset  $X$  of points defining the sets in  $\mathcal{F}'$ . Note that since  $q$  has Tukey depth greater than  $1 - \beta$  with respect to  $X$ ,  $q$  cannot be separated from  $X'$  by a hyperplane  $\Pi$ . Otherwise on the closed halfspace that contains  $q$  and that is bounded by  $\Pi$ , there are at most  $(1 - \beta)n$  points of  $X$ . Thus,  $q \in \text{Conv}(X')$ . Then by Lemma 1.5,

$$\bigcap_{x_i \in X'} H_q(x_i) = \emptyset.$$

Therefore, by the converse of Theorem 4.1 there exist an  $\alpha = c(d, 1 - \beta)$  and subfamilies  $F_1, \dots, F_{d+1}$  of  $\mathcal{F}$ , each of at least  $\alpha n$  sets, such that for every choice of sets  $H_q(x'_i) \in F_i$  we have that  $\bigcap_{i=1}^{d+1} H_q(x'_i) = \emptyset$ . Thus, by Lemma 1.5, for every such choice we have that  $q \in \text{Conv}(\{x'_1, \dots, x'_{d+1}\})$ . The result follows by setting  $X_i$  to be the set of points defining the sets in  $F_i$  and letting  $\beta$  tend to  $1 - \tau_X(q)$ .  $\square$

As with the classical versions, from the Depth Carathéodory's or Depth Helly's theorems we can prove a “depth” version of the Kirchberger's theorem.

**Theorem 4.2 (Depth Kirchberger's theorem).** *Let  $0 \leq \beta \leq 1$ ,  $d \geq 2$  and  $\alpha = c(d, 1 - \beta)$  (where  $c(d, \beta)$  is as in Theorem 1.2). Let  $R$  and  $B$  be sets of points in  $\mathbb{R}^d$  such that  $R \cup B$  has  $n$  points. Suppose that for every choice*

of subsets  $Y_1, \dots, Y_{d+2}$  of  $R \cup B$ , each of at least  $\alpha n$  points, there exists a set  $S = \{y_i : y_i \in Y_i\}$  that satisfies the following. The set  $S \cap R$  can be separated by a hyperplane from  $S \cap B$ . Then there exist subsets  $R' \subset R$  and  $B' \subset B$  such that  $|R' \cup B'| \geq \beta n$ , and  $R'$  can be separated from  $B'$  by a hyperplane.

*Proof.* We map  $R$  and  $B$  to  $\mathbb{R}^{d+1}$  as follows. Let

$$\mathcal{R} := \{(x_1, \dots, x_d, 1) : (x_1, \dots, x_d) \in R\}$$

and

$$\mathcal{B} := \{(-x_1, \dots, -x_d, -1) : (x_1, \dots, x_d) \in B\}.$$

It can be shown that a subset  $\mathcal{S}$  of  $\mathcal{R} \cup \mathcal{B}$  can be separated from the origin by a hyperplane if and only if its preimage  $S$  in  $R \cup B$  satisfies the following. The set  $S \cap R$  can be separated by a hyperplane from  $S \cap B$ .

We claim that the Tukey depth of the origin with respect to  $\mathcal{R} \cup \mathcal{B}$  is less than  $1 - \beta$ . Suppose for a contradiction the Tukey depth of the origin with respect to  $\mathcal{R} \cup \mathcal{B}$  is at least  $1 - \beta$ . By Theorem 1.2 there exist subsets  $\mathcal{Y}_1, \dots, \mathcal{Y}_{d+2}$  of  $\mathcal{R} \cup \mathcal{B}$  each of at least  $\alpha n$  points that satisfy the following. For every choice of  $\mathcal{S} = \{y_i : y_i \in \mathcal{Y}_i\}$ , the set  $\mathcal{S}$  cannot be separated from the origin by a hyperplane. Let  $Y_1, \dots, Y_{d+2}$  be the preimages of the  $\mathcal{Y}_i$ 's. Then for every subset  $S = \{y_i : y_i \in Y_i\}$ ,  $S \cap R$  cannot be separated from  $S \cap B$  by a hyperplane—a contradiction.

Therefore, there exists a subset of  $\mathcal{R} \cup \mathcal{B}$  of less than  $(1 - \beta)n$  points that can be separated from the origin by a hyperplane; the complement  $\mathcal{X}$  of this set in  $\mathcal{R} \cup \mathcal{B}$  can also be separated from the origin and has more than  $\beta n$  points. Let  $X$  be the preimage of  $\mathcal{X}$ . The result follows by setting  $R' := X \cap R$  and  $B' := X \cap B$ .  $\square$

We have used quotes when referring to the depth versions of Helly's and Kirchberger's theorems. We done so because their relationships with the notion of depth is only in their close relationship to the Depth Carathéodory's theorem. The depth versions of Carathéodory's and Helly's theorems seem to be a combination of the colorful and fractional versions. The hypothesis is that for every subfamily in which every object is assigned one of  $(d + 1)$  colors and every color class is large, there exists a colorful  $(d + 1)$ -tuple that satisfies the property. The conclusion is that a large subfamily satisfies the property. We conclude the paper by mentioning two other results that are fractional/colorful versions of their classical counterparts.

Recently, the following colorful fractional Helly's theorem has been found by Bárány, Fodor, Montejano, Oliveros and Pór.

**Theorem 4.3 (Fractional Colorful Helly's theorem [5]).** *Let  $\mathcal{F}$  be a finite family of convex sets such that each set is assigned one of  $d + 1$  colors. Let  $\mathcal{F}_1, \dots, \mathcal{F}_{d+1}$  be its color classes. Suppose that for some  $\alpha > 0$ , at least  $\alpha|\mathcal{F}_1| \cdots |\mathcal{F}_{d+1}|$  of the colorful tuples have non-empty intersection. Then some  $\mathcal{F}_i$  contains a subfamily of at least  $\frac{\alpha}{d+1}|\mathcal{F}_i|$  sets with a non-empty intersection.*

The following result was proved by Pach [27]. It is a fractional/colorful version of the Centerpoint theorem; it also bears some resemblance to the Depth Carathéodory theorem.

**Theorem 4.4.** *There exists a constant  $c_d > 0$  such that the following holds. Let  $X_1, \dots, X_{d+1}$  be finite sets of points in general position in  $\mathbb{R}^d$ . Then there exist a point  $q$  and subsets  $Y_1 \subset X_1, \dots, Y_{d+1} \subset X_{d+1}$  such that the following holds. Each  $Y_i$  has at least  $c_d |X_i|$  points; and for every choice  $x_i \in Y_i$ ,  $q$  is contained in  $\text{Conv}(\{x_1, \dots, x_{d+1}\})$ .*

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